FREE VIBRATION ANALYSIS OF A ROTATING TAPERED TIMOSHENKO BEAM

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ABSTRACT

In this study, flapwise bending vibration analysis of a tapered Timoshenko beam mounted on the periphery of a rotating rigid hub is performed. The governing differential equations of motion for pure bending are derived using the Hamilton's principle and solved using the Differential Transform Method, DTM. During the derivation of the equations, effects of rotary inertia, shear deformation and hub radius are included. The computer package Mathematica is used to write a computer program for the resulting expressions and the natural frequencies are calculated and the mode shapes are plotted. The effects of the taper ratio, the rotation speed parameter, the hub radius parameter and the Timoshenko effect parameter are investigated and the results are compared with the open literature.

Key words: Tapered beam, nonuniform beam, rotating Timoshenko beam, differential transform method

INTRODUCTION

The dynamic characteristics such as natural frequencies and related mode shapes, of rotating tapered beams are very important for the design and performance evaluation in several engineering applications including rotating machinery, helicopter blades, windmills, robot manipulators and spinning space structures. As a result, rotating tapered beams have been the subject of interest for many investigators.

In spite of their importance in engineering applications, rotating tapered Timoshenko beam problems have received less attention than rotating uniform or tapered Euler-Bernoulli beam problems. Because of the complexity of the problem, an exact solution is impossible and many approximate mathematical models have been developed to investigate the dynamic behavior of rotating tapered Timoshenko beams.

Several different techniques such as the Galerkin method, the Myklestad procedure, the finite differences approach, the perturbation technique, Bessel functions, etc.have been used for the analysis of tapered beams [14, 15, 18, 20, 21].

Different formulation and solution methods such as the matrix method, the Rayleigh-Ritz procedure, the Rayleigh-Southwell procedure, the Galerkin procedure, the finite difference scheme, the Myklestad method, the extended Holzer method, etc. have been used to consider the effect of rotation in beam [12, 13, 16, 17, 19, 22].

In this paper, the flapwise bending vibration analysis of a rotating tapered Timoshenko beam is performed by using a semi analytical-numerical technique called the Differential Transform Method, DTM. The concept of this method was first introduced by Zhou [11] in 1986 and it was used to solve both linear and nonlinear initial value problems in electric circuit analysis. The method can deal with nonlinear problems so Chiou [9] applied the Taylor transform to solve nonlinear vibration problems. Additionally, the method may be used to solve both ordinary and partial differential equations. Jang et al. [7] applied the two-dimensional differential transform method to the solution of partial differential equations. Hassan [5] adopted the differential transformation method to solve some eigenvalue problems. Since previous studies have shown that the differential transform method is an efficient tool to solve non-linear or parameter varying systems, recently it has gained much attention by several researchers [1-4].

FORMULATION

The governing partial differential equations of motion are derived for flapwise bending vibration of a rotating tapered cantilever Timoshenko beam represented by Fig. 1. Here, a cantilever tapered beam of length L, which tapers to a height h in the xz plane and which is fixed at point O to a rigid hub with radius R, is shown. The XYZ axes represent a global orthogonal coordinate system with origin at the center of mass of the hub. The beam is assumed to be rotating at a constant angular velocity Ω. In the right-handed Cartesian co-ordinate system, the X-axis coincides with the neutral axis of the beam in the

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undeflected position, the Z-axis is parallel to the axis of rotation (but not coincident) and the Y-axis lies in the plane of rotation. The principal axes of the beam cross-sections are, therefore, parallel to Y and Z directions, respectively.For this model, the beam material is homogeneous and isotropic.

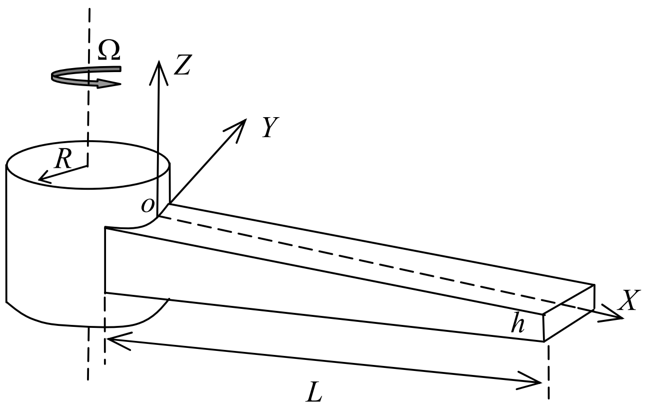

Figure 1. Configuration of a rotating Timoshenko beam that tapers in the xz plane

The Hamilton's principle is used to derive the following differential equations of motion.

$$-\rho A\frac{\partial^2 w}{\partial t^2}+\frac{\partial}{\partial x}\left(T\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}\left[kAG\left(\frac{\partial w}{\partial x}-\theta\right)\right]=0 \tag{1}$$

$$-\rho I\frac{\partial^2\theta}{\partial t^2}+\rho I\Omega^2\theta+\frac{\partial}{\partial x}\left(EI\frac{\partial\theta}{\partial x}\right)+kAG\left(\frac{\partial w}{\partial x}-\theta\right)=0 \tag{2}$$

Here ρ is the material density, A is the cross sectional area, I is the second moment of area of the beam cross section about the x axis, k is the shear correction factor, ρA is the mass per unit length, EI and kGA are the flexural rigidity and the shear rigidity of the beam, respectively.

The boundary conditions at $x=0$ and $x=L$ for Eqs. (1) and (2) are given by

$$\frac{\partial\theta}{\partial x}\delta\theta\Big|_0^L=0 \tag{3}$$

$$\left[T\frac{\partial w}{\partial x}+kAG\left(\frac{\partial w}{\partial x}-\theta\right)\right]\delta w\Big|_0^L=0 \tag{4}$$

A sinusoidal variation of $w(x,t)$ and $\theta(x,t)$ with a circular natural frequency ω is assumed and the functions are approximated as

$$w(x,t)=\overline{W}(x)e^{i\omega t} \tag{5}$$

$$\theta(x,t)=\overline{\theta}(x)e^{i\omega t} \tag{6}$$

Substituting Eqs. (5) and (6) into Eqs. (1) and (2), the equations of motion are expressed as

$$\rho A\,\omega^2\overline{W}+\frac{d}{dx}\left(T\frac{d\overline{W}}{dx}\right)+\frac{d}{dx}\left[kAG\left(\frac{d\overline{W}}{dx}-\theta\right)\right]=0 \tag{7}$$

$$\rho I\omega^2\overline{\theta}+\rho I\Omega^2\overline{\theta}+\frac{d}{dx}\left(EI\frac{d\overline{\theta}}{dx}\right)+kAG\left(\frac{d\overline{W}}{dx}-\overline{\theta}\right)=0 \tag{8}$$

Here T is the centrifugal force that varies along the spanwise direction of the beam. The expression for this force is

$$T(x)=\int_x^L\rho A\Omega^2(R+x)dx \tag{9}$$

The following equations can be used for a tapered beam

$$A(x)=A_o(1-c\frac{x}{L})^n \tag{10}$$

$$I(x)=I_o(1-c\frac{x}{L})^{n+2} \tag{11}$$

The subscript o denotes a value at the left-hand end of the tapered beam and c is a constant called the taper ratio which must be such that $c<1$ because otherwise the beam tapers to zero between its ends. Values of $n=1$ or 2 cover the most practical cases because $n=1$ gives linear variation of the area of the cross-section and cubic variation of the second moment of area along the length, whereas $n=2$ are the second and fourth orders. Thus, a large number of solids or thin-walled cross-sections can be represented by using the values, $n=1$ or $n=2$. Young's modulus E, shear modulus G and density of the material, ρ are assumed to be constant so the mass per unit length,

ρA, the flexural rigidity, EI and the shear rigidity, kAG vary according to Eqs. (10) and (11) [6].

The dimensionless parameters that are used to simplify the equations can be given as follows [8]

$$\xi = \frac{x}{L},\ \delta = \frac{R}{L},\ W(\xi) = \frac{\overline{W}}{L},\ r^2 = \frac{I_o}{A_o L^2},$$

$$s^2 = \frac{EI_o}{kA_o G L^2},\ \eta^2 = \frac{\rho A_o L^4 \Omega^2}{EI_o},\ \mu^2 = \frac{\rho A_o L^4 \omega^2}{EI_o} \tag{12}$$

Using the first two dimensionless parameters and Eq. (10), the dimensionless expression for the centrifugal force can be written as

$$T = M\left[(1-c\xi)^{n+1}(1+2c\delta+cn\delta+c\xi+nc\xi) - (1-c)^{n+1}(1+2c\delta+cn\delta+c+nc)\right] \tag{13}$$

where

$$\mathrm{M} = \frac{\rho A_o \Omega^2 L^2}{c^2 (n+2)(n+1)}$$

Substituting Eqs. (10)-(13) into Eqs. (7) and (8), the general form of the dimensionless equations of motion for any value of n are derived as follows

$$\frac{d}{d\xi}\left\{\left[(1-c\xi)^{n+1}(1+2c\delta+cn\delta+c\xi+nc\xi) - (1-c)^{n+1}(1+2c\delta+cn\delta+c+nc)\right]\frac{dW}{d\xi}\right\} + \frac{\mu^2}{\eta^2}c^2(n+1)(n+2)(1-c\xi)^n W + \frac{1}{s^2\eta^2}c^2(n+1)(n+2)\frac{d}{d\xi}\left[(1-c\xi)^n\left(\frac{dW}{d\xi}-\overline{\theta}\right)\right] = 0 \tag{14}$$

$$\frac{d}{d\xi}\left[(1-c\xi)^{n+2}\frac{d\overline{\theta}}{d\xi}\right] + r^2(\mu^2+\eta^2)(1-c\xi)^{n+2}\overline{\theta} + \frac{1}{s^2}(1-c\xi)^n\left(\frac{dW}{d\xi}-\overline{\theta}\right) = 0 \tag{15}$$

The dimensionless boundary conditions are expressed as

$$\left.\frac{d\theta}{d\xi}\delta\theta\right|_0^1 = 0 \tag{16}$$

$$\left.\left[T\frac{dW}{d\xi} + kAG\left(\frac{dW}{d\xi}-\theta\right)\right]\delta w\right|_0^1 = 0 \tag{17}$$

For a cantilever beam using Eqs.(16) and (17), the boundary conditions can be written as follows

$$\theta = W = 0 \qquad \text{at } \xi = 0 \tag{18}$$

$$\frac{dW}{d\xi} = 0 \text{ and } \frac{dW}{d\xi} - \theta = 0 \qquad \text{at } \xi = 1 \tag{19}$$

In this study the beam tapers in the xz plane. For the case where the beam tapers linearly in one plane ($n = 1$), Eqs. (14) and (15) reduces to

$$\frac{d}{d\xi}\left\{\left[(1-c\xi)^2(1+3c\delta+2c\xi) - (1-c)^2(1+3c\delta+2c)\right]\frac{dW}{d\xi}\right\} + \frac{6c^2\mu^2}{\eta^2}(1-c\xi)W + \frac{6c^2}{s^2\eta^2}\frac{d}{d\xi}\left[(1-c\xi)\left(\frac{dW}{d\xi}-\overline{\theta}\right)\right] = 0 \tag{20}$$

$$\frac{d}{d\xi}\left[(1-c\xi)^3\frac{d\overline{\theta}}{d\xi}\right] + r^2(\mu^2+\eta^2)(1-c\xi)^3\overline{\theta} + \frac{1}{s^2}(1-c\xi)\left(\frac{dW}{d\xi}-\overline{\theta}\right) = 0 \tag{21}$$

THE DIFFERENTIAL TRANSFORM METHOD

The differential transform method is a transformation technique based on the Taylor series expansion and it is a useful tool to obtain analytical solutions of the differential equations. In this method, certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem. It is different from high-order Taylor series method because Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. The differential transform method is an iterative procedure to obtain analytic Taylor Series solutions of differential equations.

Consider a function $f(x)$ which is analytic in a domain D and let $x = x_0$ represent any point in D. The function $f(x)$ is then represented by a power series whose center is located at x_0. The differential transform of the function $f(x)$ is given by

$$F[k]=\frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \quad (22)$$

where $f(x)$ is the original function and $F[k]$ is the transformed function. The inverse transformation is defined as

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F[k] \quad (23)$$

Combining Eqs. (22) and (23), we get

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \quad (24)$$

Considering Eq.(24), it is noticed that the concept of differential transform is derived from Taylor series expansion. However, the method does not evaluate the derivatives symbolically.

In actual applications, the function $f(x)$ is expressed by a finite series and Eq. (24) can be written as follows

$$f(x)=\sum_{k=0}^{m}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0} \quad (25)$$

which means that

$f(x)=\sum_{k=m+1}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)_{x=x_0}$ is negligibly small. Here, the value of m depends on the convergence of the natural frequencies.

Theorems that are frequently used in the transformation procedure are introduced in Table 1 and theorems that are used for boundary conditions are introduced in Table 2.

Table 1. Basic theorems of DTM

Original Function	DTM
$f(x)=g(x)\pm h(x)$	$F[k]=G[k]\pm H[k]$
$f(x)=\lambda g(x)$	$F[k]=\lambda G[k]$
$f(x)=g(x)h(x)$	$F[k]=\sum_{l=0}^{k}G[k-l]H[l]$
$f(x)=\frac{d^n g(x)}{dx^n}$	$F[k]=\frac{(k+n)!}{k!}G[k+n]$
$f(x)=x^n$	$F[k]=\delta(k-n)=\begin{cases}0 & if\ k\neq n\\ 1 & if\ k=n\end{cases}$

Table 2. DTM theorems for boundary conditions

$x=0$	$\frac{df}{dx}(0)=0$	$F(0)=0$
	$\frac{df}{dx}(0)=0$	$F(1)=0$
	$\frac{d^2 f}{dx^2}(0)=0$	$F(2)=0$
	$\frac{d^3 f}{dx^3}(0)=0$	$F(3)=0$
$x=1$	$f(1)=0$	$\sum_{k=0}^{\infty}F(k)=0$
	$\frac{df}{dx}(1)=0$	$\sum_{k=0}^{\infty}kF(k)=0$
	$\frac{d^2 f}{dx^2}(1)=0$	$\sum_{k=0}^{\infty}k(k-1)F(k)=0$
	$\frac{d^3 f}{dx^3}(1)=0$	$\sum_{k=0}^{\infty}(k-1)(k-2)kF(k)=0$

FORMULATION WITH DTM

In the solution step, the Differential Transform Method is applied to Eqs.(20) and (21) by using the theorems introduced in Table 1 and the following expressions are obtained. Here we remove the bar symbol from $\overline{\theta}$, and instead, we use θ.

$$c^2(k+1)(k+2)\left(3-2c+6\delta-3c\delta+\frac{6}{s^2\eta^2}\right)W[k+2]-$$
$$6c^2\left(\delta+\frac{c}{s^2\eta^2}\right)(k+1)^2W[k+1]+$$
$$\left[3c^2(c\delta-1)k(k+1)+\frac{6c^2\mu^2}{\eta^2}\right]W[k]+ \tag{26}$$
$$\left[2c^3(k-1)(k+1)-\frac{6c^3\mu^2}{\eta^2}\right]W[k-1]+$$
$$\frac{6c^3}{s^2\eta^2}(k+1)\theta[k]-\frac{6c^2}{s^2\eta^2}(k+1)\theta[k+1]=0$$

$$(k+1)(k+2)\theta[k+2]-3c(k+1)^2\theta[k+1]+$$
$$\left[3c^2k(k+1)-\frac{1}{s^2}+r^2\left(\eta^2+\mu^2\right)\right]\theta[k]-$$
$$\left[c^3(k-1)(k+1)-\frac{c}{s^2}+3cr^2\left(\eta^2+\mu^2\right)\right]\theta[k-1]+ \tag{27}$$
$$3c^2r^2\left(\eta^2+\mu^2\right)\theta[k-2]-c^3r^2\left(\eta^2+\mu^2\right)\theta[k-3]-$$
$$\frac{ck}{s^2}W[k]+\frac{(k+1)}{s^2}W[k+1]=0$$

RESULTS AND DISCUSSIONS

The computer package Mathematica is used to write a computer program for the expressions obtained using DTM. In order to validate the calculated results, comparisons with open literature are made and related graphics are plotted. The effects of the taper ratio, c, the rotation speed parameter, η, the Timoshenko effect parameter, r and the hub radius parameter, δ, are investigated.

In Table 3, variation of the first three natural frequencies of a uniform beam with respect to the Timoshenko effect, r, and the rotation speed parameter, η, are intoduced and the results are compared with the ones in Table 3 of Ref. [8] where only the fundamental natural frequencies are given. As expected, the values of the natural frequencies increase when the rotation speed parameter is increased due to the stiffening effect. The effect of the rotation speed parameter can be observed better by examining Fig. 2 where variation of the natural frequencies with respect to the taper ratio, c,and the rotation speed parameter, η, is shown. It can be noticed that in Fig. 2, the natural frequencies increase as the rotation speed parameter increases. Additionally, natural frequencies decrease as the taper ratio increases because increasing taper ratio has a softening effect resulting from the decrease of the cross-sectional area.

As can be seen from the results of Table 3, the natural frequencies decrease as the Timoshenko effect is increased. The effect of the dimensionless parameter, r, can be observed better by examining Fig. 3 where variation of the first six natural frequencies with respect to the Timoshenko effect parameter, r and the taper ratio, c is given.

When Figs. 2 and 3 are examined together, it is noticed that the nondimensional parameter, r and η have different influences on the taper ratio effects. As mentioned before, natural frequencies decrease with an increasing taper ratio. In Fig. 3, it is observed that this difference is more obvious for high order frequencies at $r=0$ (Euler-Bernoulli beam). However, this difference vanishes as the Timoshenko effect increases. However, in Fig. 2, it is observed that the rotation speed parameter does not have the same vanishing effect as the Timoshenko effect parameter.

In Table 4, variation of the first six natural frequencies of a tapered Timoshenko beam with respect to the rotation speed parameter, η, is introduced and the results are compared with the ones in Table 1 of Ref.[10]. In this study, r^2 and s^2 correspond to η and μ parameters in Ref.[10].

In Figure 4, variation of the first six natural frequencies with respect to the Timoshenko effect parameter, r and hub radius parameter, δ, is introduced. As can be seen from the results of Figure 4, the hub radius parameter has an increasing effect on the natural frequencies.

Additionally, the first four normal mode shapes of the rotating tapered Timoshenko beam whose natural frequencies are given in Table 3 are introduced in Fig. 5(a)-(d).

Table 3. Variation of the first four natural frequencies with respect to the Timoshenko effect and rotation speed parameter ($c=0$, $k=2/3$, $E/G=8/3$, $\delta=0$)

Natural Frequencies				
r	$\eta=0$		$\eta=4$	
	Present	**Ref[8]**	**Present**	**Ref[8]**
0	3.5161	3.5160	5.5850	5.585
	22.0338	-	24.2727	-
	61.6944	-	63.9640	-
0.02	3.4998	3.4998	5.5616	5.5616
	21.3540	-	23.6055	-
	57.4683	-	59.8095	-

0.04	3.4527	3.4527	5.4952	5.4951
	19.6492	-	21.9553	-
	48.8878	-	51.4808	-
0.06	3.3788	3.3787	5.3955	5.3954
	17.5467	-	19.9660	-
	40.7439	-	43.7356	-
0.08	3.2838	3.2837	5.2749	5.2749
	15.4882	-	18.0627	-
	34.3000	-	37.7312	-
0.1	3.1738	3.1738	5.1449	5.1448
	13.6606	-	16.3946	-
	29.3611	-	33.1790	-

Table 4. Variation of the first six natural frequencies of a tapered Timoshenko beam with respect to the rotation speed parameter ($c = 0.5$, $r^2 = 0.0064$, $s^2 = 0.01958$, $\delta = 0$)

Natural Frequencies	**Rotation Speed Parameters**			
	$\eta = 0$		$\eta = 3$	
	Present	**Ref[10]**	**Present**	**Ref[10]**
μ_1	3.64996	3,6500	4,88654	4,8866
μ_2	15.0218	15,0022	16,4599	16,460
μ_3	32.7840	32,785	34,4564	34,458
μ_4	53.3391	53,341	55,3555	53,358
μ_5	75.4955	-	77,8802	-
μ_6	98.1897	-	100,9190	-
Natural Frequencies	**Rotation Speed Parameters**			
	$\eta = 5$		$\eta = 10$	
	Present	**Ref[10]**	**Present**	**Ref[10]**
μ_1	6,4711	6,4712	10,9905	10,991
μ_2	18,7434	18,744	26,9280	26,928
μ_3	37,2226	37,224	47,8827	47,883
μ_4	58,7281	58,730	71,9847	71,986
μ_5	81,8834	-	97,6447	-
μ_6	105,4640	-	119,8910	-

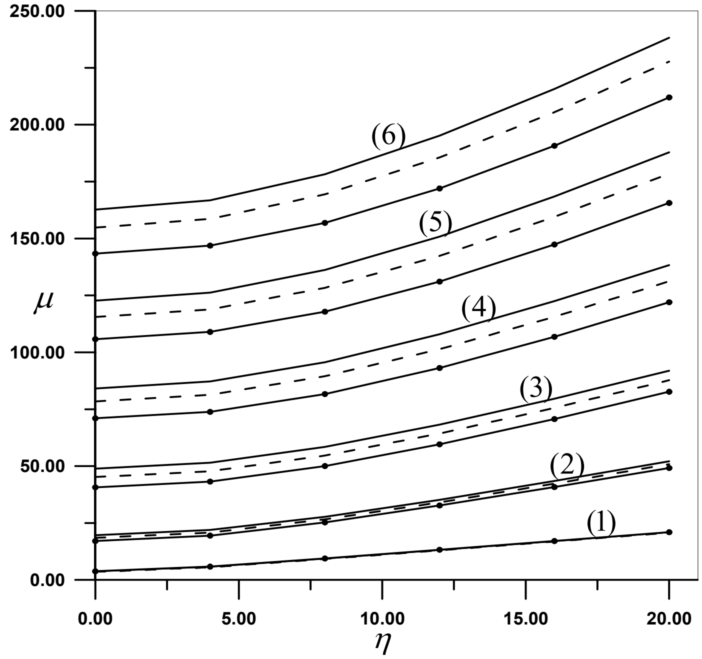

Figure 2. Variation of the first six natural frequencies with respect to the rotation speed parameter, η, and the taper ratio, c .($c = 0$, ———— ; $c = 0.25$,--------; $c = 0.5$, ●—●—●—● ; $r = 0.04$, $k = 2/3$, $E/G = 8/3$, $\delta = 0$)

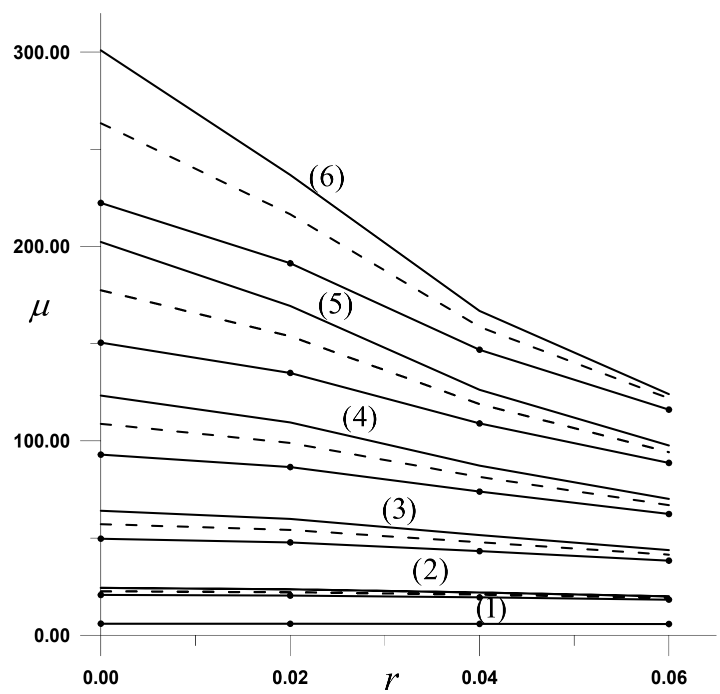

Figure 3. Variation of the first six natural frequencies with respect to the Timoshenko effect, r , and the taper ratio, c . ($c = 0$,———— ; $c = 0.25$,-------- ; $c = 0.5$, ●—●—●—● ; $\eta = 4$, $k = 2/3$, $E/G = 8/3$, $\delta = 0$)

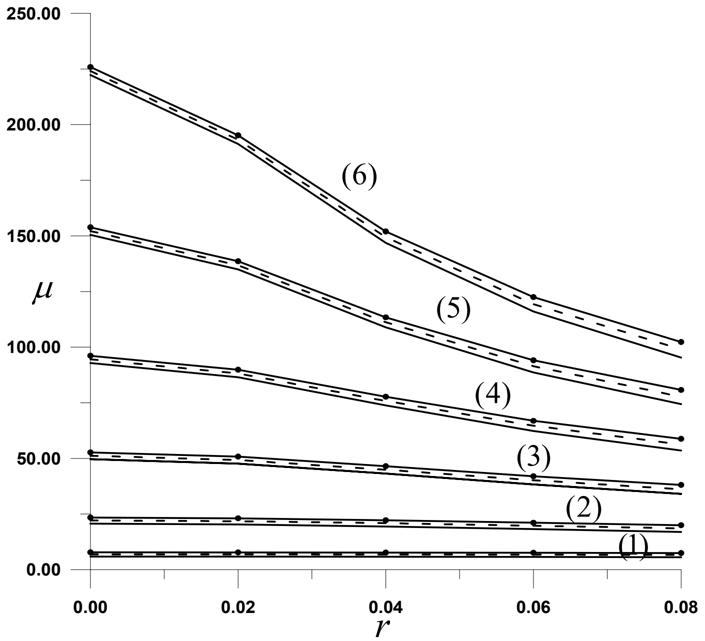

Figure 4. Variation of the first six natural frequencies with respect to the Timoshenko effect and hub radius ($c = 0.5$, $\eta = 4$, $k = 2/3$, $E/G = 8/3$, $\delta = 0$, ———; $\delta = 0.5$, - - - - - - - - ; $\delta = 1$, ●—●—●—●)

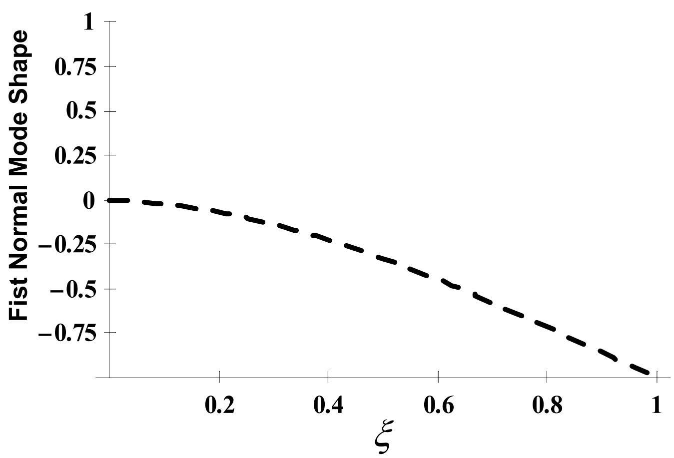

Figure 5 (a). First normal mode shape of rotating tapered Timoshenko beam ($c = 0.5$, $\eta = 3$, $r^2 = 0.0064$, $s^2 = 0.01958$)

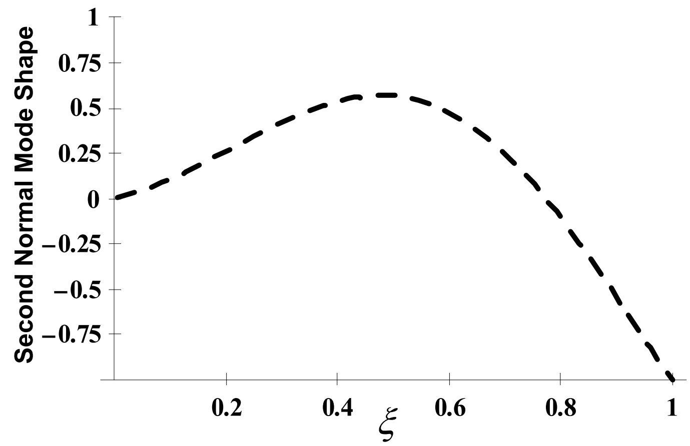

Figure 5 (b). Second normal mode shape of rotating tapered Timoshenko beam ($c = 0.5$, $\eta = 3$, $r^2 = 0.0064$, $s^2 = 0.01958$)

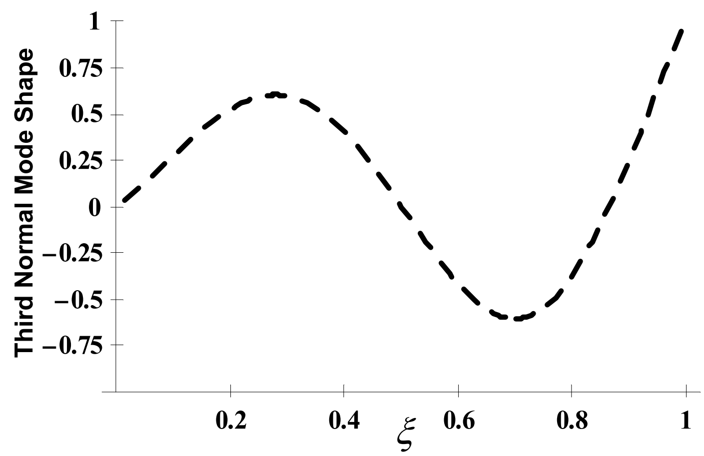

Figure 5 (c). Third normal mode shape of rotating tapered Timoshenko beam ($c = 0.5$, $\eta = 3$, $r^2 = 0.0064$, $s^2 = 0.01958$)

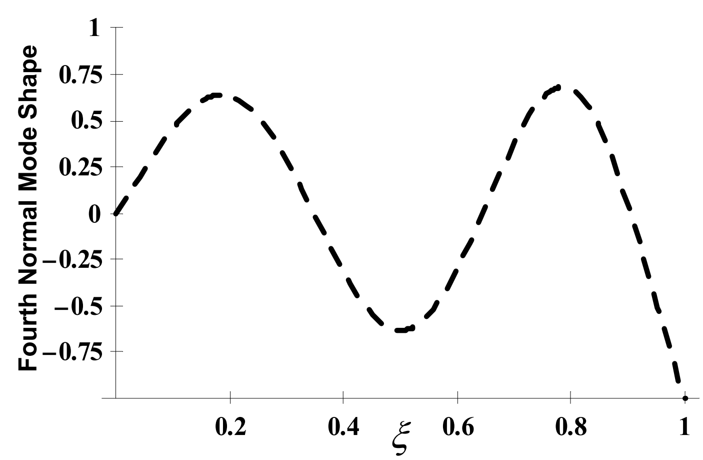

Figure 5 (d). Fourth normal mode shape of rotating tapered Timoshenko beam ($c = 0.5$, $\eta = 3$, $r^2 = 0.0064$, $s^2 = 0.01958$)

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